New symmetry current for massive spin-$\frac{3}{2}$ fields

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Abstract

We present several new results which will be of value to theorists working with massive spin-$\frac{3}{2}$ vector-spinor fields as found, for example, in low and intermediate energy hadron physics and also linearized supergravity. The most general lagrangian and propagator for a vector-spinor field in d-dimensions is given. It is shown that the observables of the theory are invariant under a novel continuous symmetry group which is also extended to an algebra. A new technique is developed for exploring the consequences of the symmetry and a previously unknown conserved vector current and charge are found. The current leads to new interactions involving spin-$\frac{3}{2}$ particles and may have important experimental consequences.

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I. GENERAL RARITA-SCHWINGER ACTION

In many instances, a relativistic field theory description of particles of spin ≥ 1 requires the presence of auxiliary lower-spin degrees of freedom. This arises from the fact that a massive field of spin s has $2s + 1$ degrees of freedom, whereas the description in terms of Lorentz covariant tensors and spinors may require more than $2s+1$ components when $s \geq 1$. Already when $s = 1$ an extra spin-0 field is needed and the situation becomes more interesting when $s \geq \frac{3}{2}$ since several extra fields are required and there appears a new symmetry of the theory. The symmetry is realized by a continuous group of transformations which redefine these lower spin auxiliary fields while leaving the observable physics unaffected. The first non-trivial case where this new symmetry group appears is that of spin-$\frac{3}{2}$ where two spin-$\frac{1}{2}$ auxiliary fields are needed. In this paper we examine some of the consequences of this symmetry. We work in d spacetime dimensions so that our expressions remain general and can be more easily facilitate dimensional regularization in hadronic applications.

The most general lagrangian for a massive vector-spinor field in d dimensions is given by [9]

$$\mathcal{L} = \overline{\psi}_\alpha \left(\Gamma^{\alpha\mu\beta} i\partial_\mu - m\Gamma^{\alpha\beta} \right) \psi_\beta, \tag{1}$$

where ψ^α is a complex vector-spinor field with suppressed spinor index [10] and

$$\begin{aligned}
\Gamma^{\alpha\mu\beta} &= g^{\alpha\beta}\gamma^\mu - A_1 g^{\mu\beta}\gamma^\alpha - A_2 g^{\mu\alpha}\gamma^\beta + A_3\gamma^\alpha\gamma^\mu\gamma^\beta, \\
\Gamma^{\alpha\beta} &= g^{\alpha\beta} - A_4\gamma^\alpha\gamma^\beta.
\end{aligned} \tag{2}$$

The coefficients are defined in terms of a complex parameter a by

$$\begin{aligned}
A_1 &= 1 + \frac{(d-2)}{d}a^*, \quad A_2 = 1 + \frac{(d-2)}{d}a, \\
A_3 &= 1 + \frac{(d-2)}{d}\left[\frac{(d-1)}{d}|a|^2 + a^* + a \right], \\
A_4 &= 1 + \frac{(d-1)}{d}\left[|a|^2 + a^* + a \right].
\end{aligned} \tag{3}$$

This can be written in a simpler way by defining the transformation $\theta^{\mu\nu}(a)$ as follows

$$\theta^{\mu\nu}(a) = g^{\mu\nu} + \frac{a}{d}\gamma^\mu\gamma^\nu, \quad a \neq -1. \tag{4}$$

Then our action (1) becomes

$$\mathcal{L} = \overline{\psi}^\alpha \theta_{\alpha\mu}(a^*) \left(\gamma^{\mu\rho\nu} i\partial_\rho + m\gamma^{\mu\nu} \right) \theta_{\nu\beta}(a)\psi^\beta, \tag{5}$$

where the totally antisymmetric combinations of gamma matrices are given by $\gamma^{\mu\rho\nu} = \frac{1}{2}\left(\gamma^{\mu}\gamma^{\rho}\gamma^{\nu} - \gamma^{\nu}\gamma^{\rho}\gamma^{\mu}\right)$ and $\gamma^{\mu\nu} = \frac{1}{2}\left(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}\right)$. In the massless limit, the action (1) has the invariance $\delta\psi^{\alpha} = \theta^{\alpha\beta}(\tilde{a})\,\partial_{\beta}\epsilon$ for arbitrary spinor ϵ, where the inverse transformation $\theta^{\alpha\beta}(\tilde{a})$ is defined below. The parameter choice $a = 0$ corresponds to the expression commonly found [2] as the massive gravitino action in linearized supergravity and in that case the invariance reduces to the usual massless gravitino gauge invariance. In addition, the choices $a = \frac{d}{1-d}$ and $a = \frac{(A+1)d}{2-d}$ for real A respectively give the original Rarita-Schwinger action [3] and an expression often seen in nuclear resonance applications. In fact, all of the vector-spinor lagrangians found in the literature are equivalent to the general action for various choices of parameter. This is as it should be since the general form of the action is dictated by the principles of quantum field theory [4]. Notice that the choice of parameter giving the supergravity action is such that the dimension of spacetime, d, does not explicitly appear. This, along with the antisymmetry of $\gamma^{\mu\rho\nu}$ and $\gamma^{\mu\nu}$, make manipulations simpler and results more transparent which is why we choose to use this parametrization of the general action.

II.   POINT TRANSFORMATION GROUP

The transformations given by (4) are sometimes called 'point' or 'contact' transformations in the literature. When $a \neq -1$ they form a group with

$$\theta^{\mu\nu}(a)\theta_{\nu\lambda}(b) = \theta^{\mu}_{\lambda}(a + b + ab) \equiv \theta^{\mu}_{\lambda}(a \circ b),$$
$$(\theta^{\mu\nu})^{-1}(a) = \theta^{\mu\nu}(\frac{-a}{1+a}) \equiv \theta^{\mu\nu}(\tilde{a}), \tag{6}$$

where we have defined the 'circle' operation $a \circ b = a + b + ab$ and also the inverse parameter $\tilde{a} = \frac{-a}{1+a}$. The transformation becomes singular at the parameter value $a = -1$ as can be seen by the fact that $k \circ -1 = -1$ for any k, so that

$$\theta^{\mu\nu}(-1)\theta_{\nu\lambda}(k) = \theta^{\mu}_{\lambda}(-1) \quad \forall\, k. \tag{7}$$

Interestingly, $a = -1$ gives the additive identity element of an algebra defined by the following addition rule

$$\theta_{\mu\nu}(a) + \theta_{\mu\nu}(b) = \theta_{\mu\nu}(a + b + 1),$$
$$\theta_{\mu\nu}(a) - \theta_{\mu\nu}(b) = \theta_{\mu\nu}(a - b - 1). \tag{8}$$

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This addition is, of course, defined *modulo zero*, where zero is the two-sided ideal generated by $\theta_{\mu\nu}(-1)$. It is easily shown that the multiplication is distributive over addition.

We should mention that one can redefine the parameter in various ways to make the group more convenient. For example, by shifting the singular point of the parameter space to $-\infty$ by letting $a \to e^{\alpha} - 1$ the group becomes [5]

$$\theta^{\mu\nu}(\alpha) = g^{\mu\nu} + \frac{e^{\alpha} - 1}{d}\gamma^{\mu}\gamma^{\nu} = e^{\frac{\alpha}{d}\gamma^{\mu}\gamma^{\nu}}$$

$$\theta^{\mu\nu}(\alpha)\theta_{\nu\lambda}(\beta) = \theta^{\mu}_{\lambda}(\alpha + \beta), \tag{9}$$

$$\left(\theta^{\mu\nu}\right)^{-1}(\alpha) = \theta^{\mu\nu}(-\alpha).$$

An even more convenient redefinition is so that the singular point is at 0. One has $a \to \alpha - 1$ and the algebra is then defined by

$$
\begin{array}{rll}
\theta^{\mu\nu}(\alpha) = & g^{\mu\nu} + \frac{\alpha-1}{d}\gamma^{\mu}\gamma^{\nu}, & \text{(definition)} \\[4pt]
\theta^{\mu\lambda}(\alpha)\theta_{\lambda}{}^{\nu}(\beta) = & \theta^{\mu\nu}(\alpha\beta), & \text{(multiplication)} \\[4pt]
\theta^{\mu\nu}(1) = & g^{\mu\nu}, & \text{(mult. id.)} \\[4pt]
\left(\theta^{\mu\nu}\right)^{-1}(\alpha) = & \theta^{\mu\nu}(\frac{1}{\alpha}), & \text{(mult. inv.)} \\[4pt]
\theta^{\mu\nu}(\alpha) + \theta^{\mu\nu}(\beta) = & \theta^{\mu\nu}(\alpha + \beta), & \text{(addition)} \\[4pt]
\theta^{\mu\nu}(0) = & g^{\mu\nu} - \frac{1}{d}\gamma^{\mu}\gamma^{\nu}, & \text{(additive id.)} \\[4pt]
\theta^{\mu\nu}(\alpha) - \theta^{\mu\nu}(\beta) = & \theta^{\mu\nu}(\alpha - \beta), & \text{(additive inv.)}
\end{array}
$$

where the addition is again defined modulo the additive identity. These redefinitions have advantages in terms of the simplicity of algebraic operations; nevertheless, for now we will continue to use the parameter as defined by (4).

The path integral is invariant under a global point transformation of the fields since the functional determinant is trivial and factors out of the integral to be cancelled out of the generating functional by the identical factor in the denominator. Hence there are no path integral anomalies and all physical correlation functions are independent of the parameter a. In the interacting theory the same will be true if one chooses the interaction to satisfy the same general criteria used in deriving the general free action: hermiticity, linearity in derivatives and non-singular behavior [4]. The full interacting theory will then have the same parameter dependence as the free theory, so that a shift of parameter in the full action is the same as a point transformation of the fields. This independence of correlation functions on

the value of the parameter can also be seen in the following way. Write the general action (5) as

$$\mathcal{L} = \overline{\psi}^\alpha \Lambda_{\alpha\beta}(a)\psi^\beta = \overline{\psi}^\alpha \theta_{\alpha\mu}(a^*)\Lambda_{\mathrm{SG}}^{\mu\nu}\theta_{\nu\beta}(a)\psi^\beta, \tag{10}$$

where $\Lambda_{\mathrm{SG}}^{\mu\nu}$ is defined by comparison with (5). The propagator Γ_{SG} (dropping indices) is the inverse of Λ_{SG}. The general propagator can be found from this by applying point transformations to the relation $\Lambda_{\mathrm{SG}}\Gamma_{\mathrm{SG}} = 1$,

$$\theta(a^*)\Lambda_{\mathrm{SG}}\theta(a)\theta^{-1}(a)\Gamma_{\mathrm{SG}}\theta^{-1}(a^*) = 1. \tag{11}$$

The general propagator is then

$$\Gamma(a) = \theta^{-1}(a)\Gamma_{\mathrm{SG}}\theta^{-1}(a^*). \tag{12}$$

Thus the propagator line in a Feynman diagram carries the inverse of the transformation on each end. Any interaction vertex which obeys our general conditions will carry a corresponding transformation to cancel it and so all observable quantities will be invariant under a shift of the parameter. It can be set to any convenient value in the quantum theory and observable physics will be unaffected.

III.   NEW SYMMETRY CURRENT

We will now examine one of the ways in which the point transformation invariance of the correlation functions can be exploited. Since the observables of the quantum theory are independent of the parameter choice, we would like to explore the consequences of this invariance using Noether's method. However, the classical action is not invariant under the transformation since the symmetry transformation, $\theta(k)$, of the field is *non-unitary*, $\theta(k^*) \neq \theta^{-1}(k)$. Under a global point transformation, $\psi_\mu \to \theta_{\mu\nu}(k)\psi^\nu$, the lagrangian (1) is not invariant, but transforms as $\mathcal{L}(a) \to \mathcal{L}(a \circ k)$. In the absence of interactions the equations of motion are invariant since the free field equations of motion imply that $\gamma \cdot \psi = 0$ and this makes the transformation (4) trivial. However, in the interacting theory this is no longer true. In order to explore the consequences of the symmetry we will therefore *impose it* on the classical action by using the following technique. We simply demand that:

- *classical actions will be considered equivalent if they lead to the same quantum theory.*

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Put in another way this says:

- *classical actions will be considered equivalent if they are related by a circle-shift of the parameter.*

This makes the point transformations a symmetry of *equivalence classes* of classical actions and we can use Noether's method to examine the consequences.

For simplicity, we will re-write our action so that it is symmetric in derivatives and we will restrict the parameter to be real. Thus

$$\mathcal{L}(a) = \overline{\psi}_\alpha \left[\frac{1}{2}\Gamma^{\alpha\rho\beta}(a)i\overset{\leftrightarrow}{\partial}_\rho + m\Gamma^{\alpha\beta}(a) \right] \psi_\beta, \tag{13}$$

where we have written

$$\Gamma^{\alpha\rho\beta}(a) = \theta^\alpha_\mu(a)\gamma^{\mu\rho\nu}\theta^\beta_\nu(a),$$
$$\Gamma^{\alpha\beta}(a) = \theta^\alpha_\mu(a)\gamma^{\mu\nu}\theta^\beta_\nu(a), \tag{14}$$
$$\overset{\leftrightarrow}{\partial}_\rho = \overset{\rightarrow}{\partial}_\rho - \overset{\leftarrow}{\partial}_\rho.$$

Under an infinitesimal local point transformation, $\theta(k(x))$, the lagrangian varies as $\mathcal{L}(a) \rightarrow \mathcal{L}(a \circ k) + \delta\mathcal{L}$ where $\delta\mathcal{L}$ contains the derivative acting on the parameter and $\mathcal{L}(a \circ k)$ is defined in exactly the same way as $\mathcal{L}(a)$ in (13) with the derivatives acting only on the fields and not on the parameter. Explicit computation gives

$$\delta\mathcal{L} = \frac{i}{2d}\overline{\psi}_\alpha \left[\Gamma^{\alpha\rho\beta}\gamma_\beta\gamma^\nu - \gamma^\alpha\gamma_\beta\Gamma^{\beta\rho\nu} \right] \psi_\nu \left(\partial_\rho k \right). \tag{15}$$

Integrating by parts and discarding the surface term we have

$$\mathcal{L}(a) \rightarrow \mathcal{L}(a \circ k) - \frac{1}{2d}\left(\partial_\rho J^\rho \right) k(x), \tag{16}$$

where J^ρ is given by

$$J^\rho = i\overline{\psi}_\alpha \left[\Gamma^{\alpha\rho\beta}(a)\gamma_\beta\gamma^\nu - \gamma^\alpha\gamma_\beta\Gamma^{\beta\rho\nu}(a) \right] \psi_\nu. \tag{17}$$

Our symmetry says that $\mathcal{L}(a) = \mathcal{L}(a \circ k)$ in the limit that $k(x)$ becomes constant. This demands that $\delta\mathcal{L} = 0$ in the limit of constant $k(x)$. Hence, from (16), we find a conserved current, J^ρ, associated to the global symmetry: $\partial_\rho J^\rho = 0$. We see from (17) that the current changes under point transformations by a circle-shift of the parameter and is therefore

invariant according to our symmetry. We can expand the $\Gamma^{\alpha\rho\beta}$ to find a simpler expression of the current as follows

$$
\begin{aligned}
J^\rho &= i(1+a)\overline{\psi}_\alpha \left[\gamma^\alpha g^{\rho\beta} - g^{\alpha\rho}\gamma^\beta\right]\psi_\beta, \\
&= i(1+a)\left[\overline{\psi}\cdot\gamma\psi^\rho - \overline{\psi}^\rho\gamma\cdot\psi\right].
\end{aligned}
\tag{18}
$$

Under a transformation, $\theta(k)$, the only change is the coefficient $(1+a) \to (1 + a \circ k)$. The conserved charge is given by

$$
Q = i(1+a)\int d^{d-1}x \left[\left(\overline{\psi}\cdot\gamma\right)\psi^0 - \overline{\psi}^0\left(\gamma\cdot\psi\right)\right],
\tag{19}
$$

which can be put in a more suggestive form by defining the spin-$\frac{1}{2}$ fields $\chi_1 = \gamma\cdot\psi$ and $\chi_2 = \gamma^0\psi^0$, so the charge becomes

$$
Q = i(1+a)\int d^{d-1}x \left[\chi_1^\dagger\chi_2 - \chi_2^\dagger\chi_1\right].
\tag{20}
$$

Since we have a new conserved current, J^ρ, we can couple a vector field such as the photon to it as follows

$$
\begin{aligned}
\mathcal{L}_{\text{int}} &= gJ_\mu A^\mu \\
&= ig(1+a)\left[\overline{\psi}\cdot\gamma\psi_\mu - \overline{\psi}_\mu\gamma\cdot\psi\right]A^\mu,
\end{aligned}
\tag{21}
$$

where g is a coupling constant. If this coupling is physically reasonable, then it should, among other things, have a measurable effect on the magnetic moment of the spin-$\frac{3}{2}$ particle. Furthermore, we also have the usual conserved vector current coming from electromagnetic gauge symmetry. This is given by

$$
j^\mu = \overline{\psi}_\alpha\Gamma^{\alpha\mu\beta}\psi_\beta = \overline{\psi}^\alpha\theta_{\alpha\mu}(a^*)\gamma^{\mu\rho\nu}\theta_{\nu\beta}(a)\psi^\beta.
\tag{22}
$$

We can use (1) to write this as

$$
\begin{aligned}
j^\mu &= \overline{\psi}_\beta\gamma^\mu\psi^\beta - A_1\left(\overline{\psi}\cdot\gamma\right)\psi^\mu \\
&\quad - A_1\overline{\psi}^\mu\left(\gamma\cdot\psi\right) + A_3\left(\overline{\psi}\cdot\gamma\right)\gamma^\mu\left(\gamma\cdot\psi\right),
\end{aligned}
\tag{23}
$$

where we have used $A_1 = A_2$ since the parameter is now real. The definitions of χ_1 and χ_2 allow us to write the charge as

$$
\begin{aligned}
Q_{\text{EM}} = \\
\int d^{d-1}x\left[\psi_\beta^\dagger\psi^\beta - A_1\left(\chi_1^\dagger\chi_2 + \chi_2^\dagger\chi_1\right) + A_3\chi_1^\dagger\chi_1\right]
\end{aligned}
\tag{24}
$$

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and we see that our new symmetry charge (20) involves only the cross terms contained in the usual electromagnetic charge (24).

The two currents (21) and (23) are separately conserved and hence we can form linear combinations of them to get other vector currents. Although both currents couple to the lower spin components of the vector-spinor field, we can modify how much influence each of these lower spins has. In fact it may be possible that, with judicious choices of couplings, one could eliminate the contribution of one or the other of the lower spins altogether by eliminating the cross term which contains both spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ components. Using this new freedom, it seems clear that the new current will have an important influence on the inconsistency problems that have been found in all interactions involving spin-$\frac{3}{2}$ fields [6, 7, 8]. Perhaps the inconsistencies can be made to cancel between the the different conserved currents so that the new symmetry may lead to further progress in that long-standing problem.

IV. CONCLUSION

We have presented the most general lagrangian and propagator for the Rarita-Schwinger field in d dimensions. These are given by equations (1) and (12) should prove useful in calculating higher loop effects in dimensional regularization as would occur in the effective resonance contribution to the imaginary part of pion scattering amplitudes, anomalous magnetic moments, and many other processes for which the $\Delta(1232)$ resonance or any other spin-$\frac{3}{2}$ particles play a significant role.

We have also explored the invariance properties of the general action under rotations of the lower spin, off-shell fields and found that this invariance implies the existence of a new conserved vector current and charge. A remarkable and important aspect of our technique is that we have used a symmetry coming from a *non-unitary* continuous group. The new current, in combination with the usual electromagnetic vector current leads to new (possibly even fully consistent) interactions involving spin-$\frac{3}{2}$ fields such as the electromagnetic couplings that we have given in (21) above, couplings to any other vector fields such as vector mesons, derivative couplings to scalar fields such as the pion, etc.

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V. ACKNOWLEDGMENTS

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